Approximations of Interplanetary Trajectories by Chebyshev Series

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Theme

A METHOD to obtain rapidly convergent series approximations of trajectories with close planetary encounters is presented. The expansions are obtained as finite Chebyshev polynomial series with numerical coefficients. A high-order solution is obtained by the Picard method of successive approximations. The series solution method may be used to solve either initial value or two-point boundary value problems, and may include nongravitational forces. A modification of the Picard method is introduced to reduce considerably the number of iterates. The method yields highly accurate and strongly convergent series solutions as explicit functions of time. The method requires less time of calculation than a more conventional method of numerical integration and provides more definitive and bounded error properties.

Content

The solution of two-point boundary value problems in Chebyshev series for nonlinear, ordinary differential equations has been discussed by Clenshaw and Norton[1] and extended by others. Carpenter has employed Chebyshev series to solve initial value and two-point boundary value problems in gravitational systems.[2,3] A lengthy bibliography on the use of Chebyshev series in the solution of differential equations is given in Ref. 4. The method presented here is an extension of the ones developed in the literature to the solution of two-point boundary value problems of interplanetary trajectories where the end-points of the trajectories are near the departure and arrival planets. Techniques are introduced to increase the accuracy and efficiency of the Picard iterative procedure.

The series solution offered here differs substantially from the matched asymptotic expansions.[5,6] The latter yield an analytical, finite series approximation, representing a class of trajectories satisfying the boundary conditions. The solution is developed in powers of the perturbing mass, with coefficients as analytical functions of boundary conditions and the time. Our series solution is a numerical, finite series approximation, representing only one trajectory satisfying the boundary conditions. The solution is developed in Chebyshev polynomial series in the time, with numerical coefficients. One advantage of our series solution is that a solution may be obtained to a very high accuracy with less effort than by the matched asymptotic expansions. Nearly any desired accuracy may be attained by merely increasing (or decreasing) the number of Picard iterates and adjusting the number of terms in the Chebyshev series. Once a solution is obtained, solutions of neighboring trajectories may be found with reduced effort by commencing a new Picard iteration using the known solution as the initial approximation, satisfying slightly different boundary conditions.

This technique may be applied to various formulations of the differential equations of motion of a space vehicle. If $\boldsymbol{\phi}$ is the force-vector, the equations of motion and the associated boundary conditions are

$$\ddot{\mathbf{r}} = \boldsymbol{\phi}(\mathbf{r}; \mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_p), \quad \mathbf{r}(t_0) = \mathbf{r}_0, \quad \mathbf{r}(t_f) = \mathbf{r}_f \tag{1}$$

where $\mathbf{r}$ is the heliocentric position-vector of the space vehicle and $\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_p$ are the position-vectors of the planets, with p being the number of perturbing planets. Dots denote derivatives with respect to the time (or other independent variables may be utilized). Here, $\mathbf{r}_0$ and $\mathbf{r}_f$ are the initial and final position-vectors of the space vehicle, and t_0 and t_f the initial and final times. The solution of a nonlinear, two-point boundary value problem may be obtained by the Picard method of successive approximations

$$\ddot{\mathbf{r}}^{(n+1)} = \boldsymbol{\phi}(\mathbf{r}^{(n)}; \quad \mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_p) \equiv \boldsymbol{\phi}^{(n)} \tag{2a}$$

$$\mathbf{r}^{(n+1)}(t_0) = \mathbf{r}_0, \quad \mathbf{r}^{(n+1)}(t_f) = \mathbf{r}_f \tag{2b}$$

for $n = 0, 1, 2, \ldots$, where $\mathbf{r}^{(n)}$ is the nth iterate of the position-vector of the space vehicle and $\mathbf{r}^{(0)}$ is some approximate, initial solution of the system of Eq. (1) that satisfies the associated boundary conditions.

The function $\boldsymbol{\phi}^{(n)}$, with reasonable assumptions on its behavior, may be approximated to any degree of accuracy by a finite Chebyshev polynomial series of the form

$$\boldsymbol{\phi}^{(n)} \cong \sum_{j=0}^{m-1} \boldsymbol{\alpha}_j^{(n)} T_j(x) \tag{3}$$

where m is the number of terms in the series, and $T_j(x)$ denotes a Chebyshev polynomial of degree j in the independent variable x, where x is a linear function of the time. The vector coefficient $\boldsymbol{\alpha}_j^{(n)}$ are determined during the nth iterate of the Picard process of Eq. (2) by evaluating the function $\boldsymbol{\phi}^{(n)}$ at selected values of the independent variable x. A set of selected values is chosen so as to render the Chebyshev polynomials orthogonal.[1] Upon double integration of the Chebyshev polynomial series to yield $\mathbf{r}^{(n+1)}$, the constants of integration are determined so that the boundary conditions of Eq. (2b) are satisfied. For each iteration, the coefficients are recalculated and the process repeated until the desired accuracy is attained.

The boundary conditions associated with the equations of motion (1) are satisfied for each iterate while the differential equations are satisfied only upon convergence of the process. This characteristic of boundary value methods leads to more advantageous convergence properties of the method as compared with many initial value methods which satisfy the differential equations for each iterate while satisfying the boundary conditions only upon convergence of the process.[7] The conditions for convergence of the Picard process defined by Eq. (2) are that the interval $[t_f - t_0]$ is sufficiently small and that the partial derivatives of each component of $\boldsymbol{\phi}$ with respect to each component of $\mathbf{r}$ are bounded in the interval. When the

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Picard process is convergent, the sequence of iterates converge only geometrically. This rate of convergence may be accelerated by using three successive values of the converging sequence to approximately determine the limit of the sequence. If S_k is the kth value of a converging sequence, the limit of the sequence, S_∞, may be approximated by

$$S_\infty \cong S_k-(S_k-S_{k-1})^2/(S_k-2S_{k-1}+S_{k-2}) \tag{4}$$

Extensive discussions and applications of Eq. (4) have been put forth and are mentioned in Ref. 4. The equation is often referred to as Aitken's formula.

Equation (4) is applied here with Chebyshev series to accelerate convergence of the Picard iteration process. The position-vectors from three successive iterates, $\mathbf{r}^{(n-1)}$, $\mathbf{r}^{(n)}$, and $\mathbf{r}^{(n+1)}$, are determined by Eqs. (2) as series in time. At the times corresponding to the set of selected values of x, the series are evaluated and the position-vectors corresponding to these times are found, producing the values: $\mathbf{r}^{(n)}(x_i)$, $i = 0, 1, 2, \ldots, m-1$. The three successive iterates of $\mathbf{r}^{(n)}(x_i)$, for each x_i, together with Eq. (4), yield values of $\mathbf{r}^{(n+2)}$, enabling the calculation of $\boldsymbol{\phi}^{(n+2)}(x_i)$ and the determination of the coefficients $\boldsymbol{\alpha}_j^{(n+2)}$. With the series for $\boldsymbol{\phi}^{(n+2)}$, two more iterates are obtained by use of Eqs. (2) yielding $\mathbf{r}^{(n+3)}$ and $\mathbf{r}^{(n+4)}$. The three vector quantities, $\mathbf{r}^{(n+2)}$, $\mathbf{r}^{(n+3)}$, $\mathbf{r}^{(n+4)}$ and Eq. (4) provide $\mathbf{r}^{(n+5)}$. This process is repeated to obtain convergence. The technique substantially accelerates convergence of the Picard process. Also, Eq. (4) may be applied directly to the coefficients of the Chebyshev series of successive iterates (as discussed in Ref. 4).

A technique for gradually increasing the number of terms in the Chebyshev series as the number of Picard iterates increases has been suggested by Clenshaw and Norton.[1] The technique allows an increase in the number of terms when successive iterates of the Picard process differ by less than the error of the series representation. Instead of increasing the number of terms gradually, as Clenshaw and Norton have done, we lengthen the series from m terms to $(2m-1)$ terms whenever an increase should be made. When m selected values are increased to $(2m-1)$ selected values, every other new selected value is also an old selected value (for more detail see Ref. 4). This allows the positions of the planets, $\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_p$, to be calculated at the times corresponding to a maximum number of selected values and saved before the Picard iterative procedure is begun. The technique improves the efficiency of the procedure markedly.

We present here an application of the method to an Earth-to-Mars trajectory. Departure is on August 11, 1973 from the Earth with arrival at Mars 128 days later. The heliocentric angle between initial and final position-vectors is 103°. In this application, the trajectory is segmented into three parts: one segment is from a near-Earth parking orbit, or from launch, to a point A, about 200,000 km from the Earth; an intermediate segment is from point A to a point B, about 300,000 km from Mars; and a third segment is from point B to a Martian fly-by, parking orbit, or to a Martian landing. The application presented here yields a Chebyshev series solution for the intermediate segment only, from point A to point B.

The equations of motion used are n-body, heliocentric, three-dimensional gravitational equations with the effects of the sun and the nine major planets included. Details of the series solution for the Earth-to-Mars trajectory are given in Table 1. Equation (4) and our series-lengthening technique are employed to improve the efficiency of the iterative procedure. The first column in Table 1 gives the iteration number n of the Picard process. The second column gives the number of terms in the Chebyshev polynomial series used during the nth iterate. The increase in the number of terms in successive rows is from m to $(2m-1)$. The third column gives the time in seconds required to calculate each iterate on a CDC 6600 computer. The last column gives the accuracy in kilometers of the last iterate of each row in Table 1. The accuracy is the maximum difference between the true solution and the series approximation, over the entire trajectory. The process is initiated with

Table 1 An Earth-to-Mars Chebyshev series approximation

Iteration step number	Number of terms in series	Calculation time per step, sec	Accuracy attained km
1–7	17	0.03	8.4×10^4
8–9	33	0.09	7.9×10^3
10–13	65	0.28	4.6×10^1
14–22	129	0.88	1.0×10^{-1}

rectilinear (straight line) motion from $\mathbf{r}_0$ to $\mathbf{r}_f$, the initial and final position-vectors, obviating the need to solve Lambert's theorem. The total number of iterates is 22 and the number of terms required in the series is 129 in order to attain a uniform accuracy of the solution of 100 m. The total time of calculation required by the method to obtain the solution is 16 sec. Applications to a different Earth-to-Mars trajectory and to an Earth-to-Venus trajectory are presented in Ref. 4. The Earth-to-Mars trajectory in Ref. 4 includes a close approach to Mars and is receding from Mars at the final time, whereas the trajectory presented here in Table 1 is still approaching Mars at the final time.

An eighth-order, Runge-Kutta method of numerical integration with variable step-size is applied to solve the same two-point boundary value problem. The integration method simultaneously integrates the motion of the space vehicle, the planets, and the linear variational equations of the space vehicle. Differential correction of the initial velocity is repeated until the final position is within 100 m of the desired point. This method of solution requires approximately 153 sec of calculation time. If the motions of the planets are known beforehand and not determined along with the motion of the space vehicle, the time of calculation is about 40 sec.

Some of the advantages of the method offered here are: 1) the series solution is an explicit function of time; 2) due to the properties of the Picard method of successive approximations and of the Chebyshev polynomials, the series solution has a known and bounded error with the bound being a minimum compared with other polynomial approximations (see Ref. 4); 3) the method is a boundary value method and hence converges for a larger class of boundary value problems than do initial value methods (as discussed in Ref. 7); and, 4) the method is faster in time of calculation than the particular numerical integration technique and initial value method used here to solve the same two-point boundary value problem.

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